

Tidal Effects on the Spatial Structure of the Local Group

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Abstract. AIMS. The spatial distribution of galaxies in the Local Group (LG) is the footprint of its formation mechanism and the gravitational interactions among its members and the external massive galaxies or galaxy groups. Recently, Pasetto & Chiosi (2007), using a 3D-geometrical description of the spatial distribution of all the members of the LG (not only the satellites of the MW and M31) based on present-day data on positions and distances, found that all galaxies (MW, M31, their satellites, and even the most distant objects) are confined within a slab of about 200 kpc thickness. Examining how external galaxies or groups would gravitationally affect (and eventually alter) the planar structure (and its temporal evolution) of the LG, they found that the external force field acts parallel to the plane determined by geometry and studied this with the Least Action Principle.

METHODS. In this paper, we have thoroughly investigated the role played by the tidal forces exerted by external galaxies or galaxy groups on the LG galaxies (the most distant dwarfs in particular) in shaping their large scale distribution. The idea based on the well known effect of tidal interactions, according to which a system of mass-points can undergo not only tidal stripping but also tidal compression and thus become flatter.

RESULTS. Excluding the dwarf galaxies tightly bound to the MW and M31, the same tidal forces can account for the planar distribution of the remaining dwarf galaxies. We analytically recover the results of Pasetto & Chiosi (2007) and prove that a planar distribution of the LG dwarf galaxies is compatible with the external force field. We also highlight the physical cause of this result.

Key words. Local Group, Milky Way, Andromeda, dwarf galaxies

1. Introduction

Over the years, many attempts have been made to find the spatial distribution of the LG galaxies. Limiting ourselves to a few pioneering studies and a few contributions in the past decade, Kahn & Woltjer (1959) and Raychaudhury & Lynden-Bell (1989) suggested a planar distribution based on studies of the LG dynamics; Hartwick (2000) found a flat ellipsoid with axial ratios $(a, b, c) = (1.00, 0.51, 0.19)$ which is not too different from a plane; finally Kunkel (1979), Grebel et al. (1999) and Fusi Pecci et al. (1995) suggested that the satellite dwarf galaxies of the MW and M31 lie on planes ¹.

¹ Closely related to this problem is the issue of the anisotropic distribution of inner sub-haloes with respect to larger haloes in relation to the Holmberg effect (Holmberg 1969) with dissimilar results, e.g. (Sales & Lambas 2004; Yang et al. 2006). What matters here (and is still debated) is whether disruptions and tidal effects can create the apparent polar alignment of the dwarf satellites around the host galaxy

More recently, Kroupa et al. (2005) and Metz et al. (2007) suggested a planar distribution of the satellites of the MW, which however could also be explained as a consequence of the distribution of sub-haloes (Zentner et al. 2005) in the early cosmological stages (Kang et al. 2005). The same problem has been investigated by Koch & Grebel (2006) for the satellites of M31 with similar conclusions.

Starting from the basic idea that an off-center hydrodynamical collision occurred some 10 Gyr ago between the primordial gas-rich M31 galaxy and the MW, and compressed the halo gas to form all the LG dwarf galaxies, Sawa & Fujimoto (2005) suggested that the new-born dwarf galaxies would be located near the orbital plane of the MW and M31. They argued that this view is also sustained by the visual inspection of the 2D sky distribution

or, for the particular case of the LG, the position of the dwarf galaxies is the consequence of peculiar directions of pre-existing cosmological filaments.

of the LG members and that a well-defined plane of finite thickness is found, within which most of the member galaxies are confined.

Pasetto & Chiosi (2007, hereafter PC07) attacked the problem from a completely different perspective. In summary, adopting known data on positions and distances and making use of analytical geometry, they looked for the plane that minimizes the distances of all galaxies in the LG to it (not only the MW and M31 and their satellites, but also the distant dwarfs). The second part of their study was to find a dynamical justification for the planar distribution. To this aim, they applied the Hamilton Method (Minimum Action) to investigate the dynamics of the LG complex and the action of the gravitational forces exerted by external nearby galaxies or groups. They found that the planar distribution is fully compatible with the minimum action and that the external force field is likely compatible with the plane. Such a field pulls the LG galaxies along, without altering their planar distribution. Special care was taken to evaluate the robustness of the result.

To somehow account for the different results obtained by Kroupa et al. (2005), Koch & Grebel (2006), and Sawa & Fujimoto (2005), it is worth recalling here that the various studies did not use the same galaxy sampling, start from the same working physical hypotheses nor deal with the same dynamical regime. In brief:

(i) The planes for the MW and M31 satellites (Kroupa et al. 2005; Koch & Grebel 2006, respectively) are of a very local nature as they are the consequence of strong collisional dynamics with the host galaxy (hereafter HG). No easy explanation can be found to secure that these planes will survive for long time (more than a few dynamical time scales) due to the peculiar proper motions of the dwarfs that determine these planes: see, e.g., the ideas in Lynden-Bell (1983); Lynden-Bell & Lynden-Bell (1995); Palma et al. (2002) applied then by Metz et al. (2008). What would be role of distant dwarfs in determining the structure of the whole LG is simply not considered. Therefore, these planes cannot be extended to the whole LG.

(ii) In Sawa & Fujimoto (2005) the solution for a common plane is based on the ad hoc initial hypothesis concerning the origin of the angular momentum. The sample of dwarf galaxies used to determine the plane is limited to the satellites of the two HGs. Finally, the orbits of these dwarfs are constrained to lay on this plane.

(iii) In PC07, the common plane is chosen by assuming that it contains the MW and M31 and minimizing the distances of all remaining LG galaxies to this plane, including also the distant ones. This can be justified by considering that the HG satellites are strongly influenced by local dynamics (with continuous modification of their orbits, including possible captures by the HGs) and that if a common planar distribution for all LG galaxies exists, this should be brought into evidence by the more external galaxies. They are much less likely to be affected by strong interactions with one of the HGs and therefore more sensitive to the influence of external galaxies and/or groups.

The plane found by PC07 is actually a slab of about 200 kpc thickness, i.e. it is worth mentioning that the largest apo-center of the HG satellites most probably falls inside this slab.

In addition, PC07 have shown that the external force field runs parallel to their plane. It is likely that among the galaxies of the LG those that feel the external action the most are the dwarfs not tightly bound to any HG. *In other words, this group of dwarfs could act as a tracer of the external force field. Our aim is to find a gravitational action that is able to induce, on a long time scale, a sort of extended slab. Tidal forces are known to engender this kind of response.*

The main goal of this study is to highlight the physical nature of the results obtained by PC07 that had a rather complicated dynamical description requiring a numerical approach. We therefore develop here a simpler linear approximation that is much easier to handle and yet able to provide a physical insight.

The plan of the paper is as follows. In Section 2, we define the tidal force field acting on the whole LG. The tidal forces are those developed by external groups of galaxies. We lump together the MW and its satellites (M31 and its satellites) for which a suitable treatment is required and look at the remaining dwarf galaxies. On a long time scale the external tidal forces can engender a planar distribution of these dwarf galaxies of the LG. In Section 3 we go deeper into this issue and check whether the planar distribution is a mere coincidence or the consequence of fundamental laws of mechanics. The answer is the latter: the planar distribution corresponds to a minimum energy and stable configuration of the whole system. The dwarf galaxies must lie on a plane as a consequence of the long time scale influence of the tidal forces exerted by massive galaxies or galaxy groups external to the LG. Furthermore, the plane found by PC07 and the minimum energy plane are coincident and the situation is stable. Finally, in Section 4 we summarize the results and present some general considerations.

2. LG structure and tidal forces

Looking at the composition of the LG, three main components can be identified: two massive galaxies (MW and M31), their respective groups of bounded satellites, a large number of distant dwarf galaxies loosely interacting or even uncoupled to the dominant galaxies. As far as the gravitational interaction is concerned, there are two questions we are interested in addressing:

- (1) How does the external force field change with time?
- (2) Have the dwarf galaxies that are not members of the MW or the M31 family been affected by the tidal interaction with the external force field?

First, to get a rough estimate of the influence of the tidal forces acting on the LG, we must develop an accurate geometrical description of the LG during its temporal evolution. Although the tidal forces are weak, they can

Table 1. External galaxy groups gravitationally influencing the LG dynamics. Each group is indicated by the dominant galaxy. The members of each group are the same as in PC07 to whom the reader should refer for all details (their Section 4 and Table 2). To this list the MW and M31 are added. The radial velocities are quoted relative to the center of the MW. No correction for the motion of the Sun toward the Local Standard of Rest is applied because it falls below the accuracy adopted in this study. The uncertainties on the distances and radial velocities are omitted.

Group Name	l °	b °	d Mpc	M $10^{12} M_{\odot}$	V_r km s^{-1}
IC 342	138.2	10.6	3.3	12.6	171.0
Maffei	136.4	-0.4	3.5	6.3	152.0
M 81	142.1	40.9	3.7	1.6	130.0
Cen A	309.5	19.4	2.7	4.7	371.0
Sculptor	105.8	85.8	3.2	6.3	229.0
M 83	312.1	25.9	4.5	0.8	249.0
Andromeda	121.21	-21.60	0.76	3.16	-123.00
Milky Way	0	0	0.00	2.20	0.00

produce a cumulative effect of compression that could explain the planar distribution by acting over a very long time. However, if this action is not always pointing in the same direction, the net effect can be small, and the tidal effect can no longer be the physical mechanism compressing the dwarf galaxy distribution. To follow the direction of a tidal field, we apply the usual formalism of the tidal tensor (e.g. Misner et al. (1973), Chap. 1) to all galaxies of the the LG and try to understand its global behavior under the action of the external force field.

Frame of reference. The actual mutual interaction between the MW and M31 suggests that they can be considered as a privileged system whose center of mass (CM) can be assumed as the origin of a reference frame (presented in Fig. 1 and described in more detail below). The positions and motions of any other dwarf of the LG that is not a member of the MW and M31 families can be given relative to this CM. This is reminiscent of the geometry of the restricted 3 body problem (e.g. Szebehely (1967)) but here the CM will be followed in its time evolution. In particular the stationary action principle has been applied in PC07 to produce a possible solution for the motion of the external groups acting gravitationally on the LG as well as MW and M31 (their table 4). From this solution we can infer the spatial evolution in the time, t , of the entire LG-barycentric system $\mathbf{x}^{\text{CM}_{\text{LG}}} = \mathbf{x}^{\text{CM}_{\text{LG}}}(t)$, which is nearly coincident with the center of mass of the MW + M31 system.

Here, we start by considering the general expression for the external tidal tensor due to any potential Φ expressed in the reference system S_0 of Figure 1, to express the external tidal tensor acting on the complex MW+M31

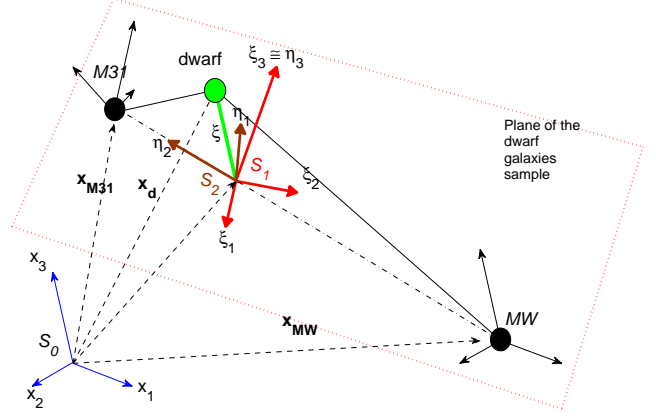


Fig. 1. Sketch of the geometrical framework we have adopted. First we define the inertial reference frame, always named S_0 , with axis (S_0, x_1, x_2, x_3) . It is centered on the CM of the external galaxy groups listed in Table 1. We then introduce two auxiliary reference frames: the first centered on the barycenter of the MW and M31, which is aligned with the principal axes of the eigen-system provided by Eqn. (1). This system is called S_1 and it has axes (ξ_1, ξ_2, ξ_3) with ξ_3 pointing roughly in the direction orthogonal to the plane of the dwarf galaxies (red dotted rectangle). The second is called S_2 and it has axes (η_1, η_2, η_3) with η_3 pointing in the same direction of ξ_3 and η_2 pointing in the direction of M31; see the text for details. The current position vector of a generic dwarf galaxy in the system S_1 and S_2 is called ξ and η respectively (e.g. we show ξ in the figure). It is always oriented toward the generic dwarf galaxy.

$$\left(\frac{3 \left(x_i^{\text{CM}_{\text{LG}}} - x_i^g \right) \left(x_j^{\text{CM}_{\text{LG}}} - x_j^g \right)}{\left\| x_i^{\text{CM}_{\text{LG}}} - x_i^g \right\|^2} - \delta_{ij} \right) \quad (1)$$

where G is the gravitational constant, x^g are the coordinates of the external galaxy groups of Table 1 of mass M_g , δ is the Dirac's function, $\|\cdot\|$ is the standard norm. We evaluate the tidal tensor at the barycenter of the LG, $\mathbf{x}^{\text{CM}_{\text{LG}}} = \mathbf{x}^{\text{CM}_{\text{LG}}}(t)$ as defined above but where the time dependence has been omitted.

To get an idea of the effects we are looking for, let us make the following preliminary considerations. Let us assume that the most distant dwarf tat define the plane π discovered by PC07 owe their distribution to the initial conditions determined by the tidal tensor of external objects. Then we expect the total tidal tensor (TTT), $T_{ij}^{\text{tot}}(\mathbf{x})$, defined by Eqn. (1) including in the sum also the MW and M31, to give, if reduced to its normal form, the most negative eigenvalue. Between the three eigenvectors of TTT, the one corresponding to this most negative eigenvalue will indicate the direction of the tidal compression (see classical text-books such as Misner et al. (1973) Chap 1, Binney & Tremaine (1987) Chap 7, or some applications as in Raychaudhury & Lynden-Bell (1989)). Thus

$$T_{ij}(\mathbf{x}^{\text{CM}_{\text{LG}}}) = \sum_{g \neq \text{MW}, \text{M31}} \frac{GM_g}{\left\| x_i^{\text{CM}_{\text{LG}}} - x_i^g \right\|^3} \times$$

Table 2. This table shows the temporal evolution of the eigenvalues and eigenvectors as a function of the look-back time in Gyr (column 1). The three eigenvalues are in columns (2) to (4) and the corresponding eigenvectors are in columns (5) through (13). The headers are self explanatory. We have highlighted the most negative values (column 4) and the corresponding eigenvectors (column 11 through 13) for all values of the look-back time at which the most negative eigenvalue keeps its sign.

t_{lb}	λ_1	λ_2	λ_3	$n_x^{(\lambda_1)}$	$n_y^{(\lambda_1)}$	$n_z^{(\lambda_1)}$	$n_x^{(\lambda_2)}$	$n_y^{(\lambda_2)}$	$n_z^{(\lambda_2)}$	$n_x^{(\lambda_3)}$	$n_y^{(\lambda_3)}$	$n_z^{(\lambda_3)}$
13.3	0.0125	-0.0049	-0.0076	0.310	0.507	0.804	0.934	-0.008	-0.356	0.174	-0.862	0.476
12.3	0.0058	0.0001	-0.0060	-0.266	0.411	0.872	0.944	0.292	0.151	0.192	-0.864	0.465
11.0	0.0031	0.0008	-0.0038	0.941	0.052	-0.334	0.265	0.498	0.826	0.209	-0.866	0.455
9.5	0.0030	-0.0002	-0.0028	0.971	0.235	-0.043	-0.066	0.436	0.897	0.230	-0.868	0.439
7.9	0.0030	-0.0006	-0.0024	0.954	0.297	0.026	-0.146	0.390	0.909	0.259	-0.871	0.416
6.2	0.0030	-0.0007	-0.0023	0.939	0.341	0.045	-0.171	0.346	0.922	0.299	-0.873	0.384
4.6	0.0033	-0.0008	-0.0024	0.919	0.391	0.050	-0.175	0.292	0.940	0.353	-0.873	0.336
3.0	0.0037	-0.0009	-0.0028	0.887	0.458	0.059	-0.171	0.208	0.963	0.429	-0.864	0.263
1.4	0.0046	-0.0010	-0.0036	-0.829	-0.551	-0.097	-0.158	0.065	0.985	0.536	-0.832	0.141
0.0	0.0065	-0.0017	-0.0047	-0.738	-0.657	-0.151	-0.064	-0.154	0.986	0.671	-0.738	-0.071

if we find this behavior also in our TTT evaluated at the position of a dwarf galaxy far away from the MW or M31 this could hint that the compression does indeed occur. For example a simple case would be that a dwarf presently belonging to the plane π was also formed by some mechanism in the plane π or close to it. In such a case we can simply make the hypothesis that the tidal force acting on this dwarf should be similar to the tidal force that acted on the plane in the past, say 9 Gyr ago. We can estimate these eigenvalues and their eigenvector directions by combining the equations of the plane π (see Eqn. (2)), obtained by looking at the current dwarf galaxy distribution, and Eqn. (1) which can also be evaluated backwards in time. If we find compatible values between different points at different epochs, then we can claim that the effect we are searching for could effectively have acted.

If the plane π is a slab with a diameter of 4 Mpc and 200 kpc thick, we can evaluate the TTT at any point on this plane $P \in \pi$, say 2 Mpc away from the barycenter of the LG, $T_{ij}^{tot}(\mathbf{x}^P)$. The resulting eigenvalue of this tidal tensor, e.g. 9 Gyr ago, is $\lambda_9^{P_{Gyr}} = \{0.024, -0.016, -0.007\}$. The same evaluation can then be repeated for the barycenter position $T_{ij}^{tot}(\mathbf{x}^{CM_{LG}})$, obtaining $\lambda_9^{CM_{LG}} = \{0.022, -0.014, -0.007\}$. This result strongly suggests that back in the past the force determining the subsequent orbital evolution of a generic dwarf had a component squeezing the motion toward the plane. Proceeding in this way we can prove the compatibility of the eigenvalues of the TTT for every position on the plane π , i.e. $T_{ij}^{tot}(\mathbf{x}^{CM_{LG}}) \cong T_{ij}^{tot}(\mathbf{x}^P) \forall P \in \pi$. This clearly allow us to explore the possibility that the plane π of PC07 is the consequence of the tidal forces acting on the LG during a large fraction of the Hubble time, i.e. we want to extend this estimation not only to the present time $t = t_0$ but also to the time² $t < t_0$.

² Of course a more correct computation could have been performed by knowing distribution of the dwarf galaxies in the past, but unfortunately we cannot track back the past orbits of the dwarf galaxies belonging to the plane π today; therefore

To proceed further, we need to search the eigenvectors associated with the tidal tensor with the most negative eigenvalues. In the limits of our approximation, they should keep a direction with respect to an inertial reference frame not too far from the normal to the geometrically plane $\hat{n}(\pi)$ for most of the Hubble time, which can nowadays be inferred by simple inspection of the dwarf galaxies' distribution in the LG. To prove this, we solve the eigen-system for the tidal tensor of Eqn. (1) as a function of time. The solutions are presented in Table 2. The results we are interested in are limited in time to a range where monotonicity of the trend of the eigenvalues can be exploited in order to reveal an integrated cumulative effect of compression or expansion. We find that our range of interest has to span the last 9 Gyr, imposing a lower limit to our analysis of $t = t_{inf} \cong 9$ Gyr. Before this t_{inf} the configuration of the eigenvalues is slightly different. From Table 2 we see that the time evolution of the eigenvalues shows a phase with one compressive direction compared to two positive expansion directions, and before that, as well as after $t = t_{inf}$, we see two negative directions compared with one positive. For simplicity we will not be treating analytically in the following sections these switchings between the configurations. We are only interested in the last most dominant time evolution of the monotonic behavior of the eigenvalues. As we will exclude the first three rows of Table 2 (the primordial evolution prior to t_{inf}) from our analysis, from now on we will uniquely identify (unless otherwise specified) with $\lambda_1, \lambda_2, \lambda_3$ the positive eigenvalue, the second negative eigenvalue and the most negative eigenvalue respectively for the eigenvectors of the external tidal tensor defined in equation (1) followed in their time t evolution exclusively for $t \in [t_{inf}, t_0]$. The most negative eigenvalue (column 4) and its evolution during the past 9 Gyr and projections of the associated eigenvector onto the axes of the inertial system are highlighted in *ital-*

Eqn. (2) for the plane π cannot be directly determined in its time evolution.

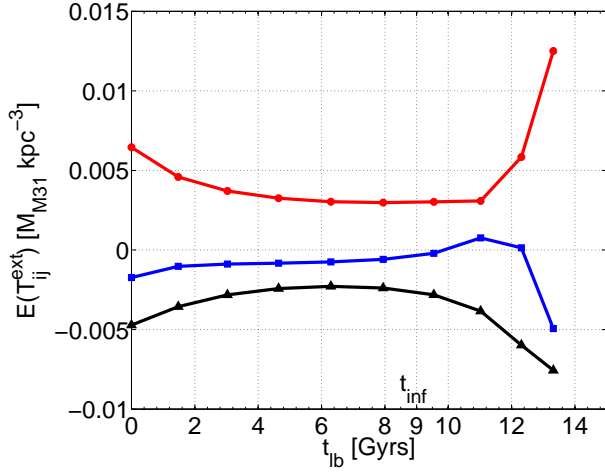


Fig. 2. Evolution of the eigenvalues $E(T_{ij}^{\text{ext}})$ of Table 2 as a function of the look-back time in Gyr (column 1). The red line with dots is for the first positive eigenvalue λ_1 , the blue line with squares is the second eigenvalue λ_2 negative for $t_{lb} < 9.5 \text{ Gys}$, and the black with triangles is for the third (most negative) eigenvalue λ_3 .

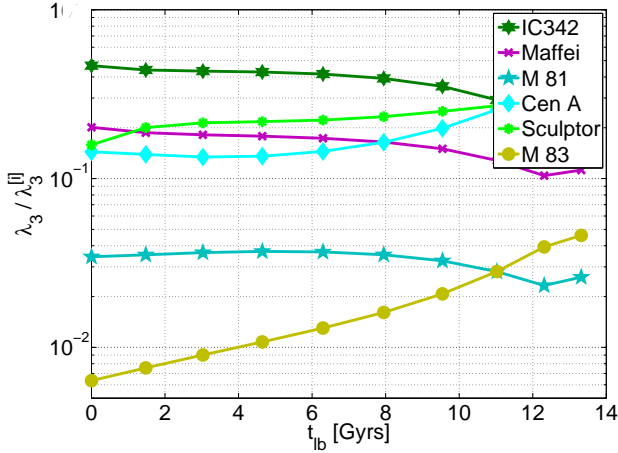


Fig. 3. Here we tracked back the influence of each individual component in the sum of Eqn. (1) for the most negative eigenvalue, λ_3^{ij} . The curves are normalized to the black line, λ_3 , of Fig. 2 (see text for details). The major contribution overall is dominated by the IC342 group, followed by Cen A, Maffei and Sculptor groups. The contribution of M83, even if nowadays marginal, could have played a role in the past in the imprint of the initial condition of the LG dwarf galaxies.

ics. The time variations of the three eigenvalues are shown in Fig. 2.

It is soon evident from the data displayed in Fig. 2 that one of the negative eigenvalues dominates. Therefore, we are in a situation in which the external field acting on the LG may engender a planar distribution of the dwarf galaxies. This is an interesting result because

1. It proves that, using the tidal tensor, we can analytically obtain the same results of PC07 for the behavior of the external field. The external force field turns out to be compatible with a flat spatial distribution of the dwarf galaxies that remains constant for a large fraction of the Hubble time. Moreover, we can argue that the external force field started to flatten the spatial distribution of the dwarf galaxies already prior to t_{inf} . If we investigate in more detail the relevance of the different groups on this flattening effect, we can plot in Fig. 3 the normalized trend of the most negative eigenvalue of Eqn.(1) split into its components. Here we see the most negative eigenvalues of the sum in Eqn.(1) normalized to the overall sum (hence the black line of Fig. 2 is here the unitary constant upper bound of the figure). As we can infer from the figure, the influence of IC342 group has always been the most significant, followed by the effects of Maffei, Sculptor and Cen A groups. This is expected from their masses and positions listed in Table 1. The time evolution shown in this Figure confirms their relative importance in the compressing effect on the LG for its temporal evolution. Slightly less important is the contribution of the M83 group that was nevertheless as important as that of the M81 group 11 Gyr ago.
2. We must clarify once and for all that the present result does *not* prove that the spatial distribution of dwarf galaxies in the LG has to be flat, but only that the external force field is compatible with such a flat distribution.
3. The fact that the external force field acting only on the two main HGs (MW and M31) of the LG is compatible with a flat spatial distribution of the distant dwarfs does not tell us anything about the distribution of the nearby dwarf satellites around their HG (both MW and M31).
4. Furthermore, the planar distribution shown by the geometrical analysis made by PC07 cannot *a priori* be related to the planar distribution suggested by the compression effect described by the tidal tensor affecting the LG. The subject of the following analysis is to explain the coincidence claimed by PC07 between the geometrical and dynamical planes.

Now we seek to prove that the following two issues are tightly related: (i) There exists a plane in the spatial distribution of the LG dwarf galaxies that is expected from the external tidal force acting on the LG. (ii) In the context of the linear approximation that we have adopted (see below), the tidal force field compatible with a flat distribution has an orientation whose normal vector is compatible with the normal vector of the planar distribution π found by PC07.

The equation for the plane π can be rewritten here in the reference frame S_0 as

$$0.64x_1 - 0.61x_2 - 0.45x_3 = 0 \quad (2)$$

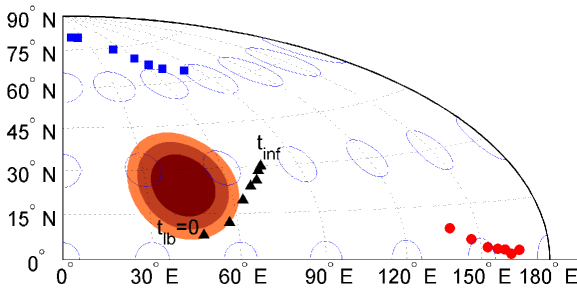


Fig. 4. The Hammer projection of the sky position of the normal to the plane found by PC07 (from purely geometrical arguments). The color code from light red to dark brown corresponds to the 3σ , 2σ , and 1σ uncertainties as estimated by PC07 from the principal component analysis and projected onto the sky. The shape of the three shaded areas becomes more oblate at increasing angular direction b . The classical Tissot's circles, perfect circles of angular radius of 7 deg have been plotted to help visualize the angular distances. We have used the same symbols and time intervals as in Fig. 2 and Table 2. The black triangles nearly overlap the shaded area, showing that the direction of the eigenvector with the most negative eigenvalue lies close to the direction of the normal to the PC07 plane, which is of course constant in the figure because it is derived from the observational data available today. For comparison we also show the evolution of the other two orthogonal eigenvectors indicated by the same symbols as in Figure 2.

with a direction $(l, b) = (45, 27)$ for $\hat{n}(\pi)$, the normal vector to the plane. Furthermore, the directions of the three basis vectors \hat{e}_{S_0} in which the eigenvalues of Table 2 are expressed have been chosen as collinear with those of the reference frame adopted by PC07 (the approximation was made such that the Sun is placed at the center of MW, the error of ≈ 8.5 kpc is negligible in the present context). Therefore, we can assume that *only* at the present time t_0 does the reference system in which the above equation of the plane is written have orthonormal vectors parallel to the basis vector of reference system adopted at $t = t_0$. It follows that the Hammer projection from the center of mass of the LG shows the eigenvector directions as a function of time and the normal vector to the geometrical plane of PC07 are as in Fig. 2.

Interestingly, it is the eigenvector relative to the most negative eigenvalue that lies closer to the direction of the normal of the plane of the dwarf galaxies.

Considering that every point in this map has an error radius of about $\pm 9^\circ$, inherited from the Minimum Action analysis, and considering the uncertainty in the angular

definition of the direction of $\hat{n}(\pi)$, the normal to the geometrical plane and the direction of the vector associated with the most negative eigenvalue are compatible at the present time at the 3σ -level of confidence. Moreover, the eigenvector of the most negative eigenvalue λ_3 of the tidal tensor moves, maintaining a direction not so far from the direction that we can nowadays deduce from the observation for $\hat{n}(\pi)$. This result, which was already present in PC07, is recovered here in a semi-analytical treatment of the whole problem. *The key question to be answered now is whether the coincidence is "causal or casual" and what the meaning of all this is.*

3. Causal or casual?

The tidal tensor can be derived from the Taylor expansion of the force field (see next section). This implies that the object we want to investigate (a dwarf galaxy) is under the effect of a smoothly varying potential in the course of evolution.

The large scale description of the gravitational interaction adopted here and by PC07 does not work at the distance scales of the closer HG-dwarf satellite interactions where satellite dwarf galaxies suffer kick-off, are continually absorbed into the halo of the HG (MW and M31 in our case) and the collisionless description is not correct. A dwarf galaxy of these closer samples, undergoing the much more intense direct interaction with the HG, does not significantly respond to the weaker external field acting on it. On the other hand, the collisionless description is suited to deal with the effects of distant galaxy groups (see the list in Table 1) and hence the influence of the external field on the other dwarf galaxies far away from a HG (see e.g. Raychaudhury & Lynden-Bell (1989); Peebles (1990); Dunn & Laflamme (1993); Peebles (1994)).

The frame of reference. In order to formalize what is in the previous section we present the following two hypotheses, one on the directions of the eigenvectors and the other on their values:

1. We will not consider the evolutionary stages of the Universe older than $t_{lb} > 9$ Gyr (look-back time) to work with a monotonic behaviour of a single negative eigenvalue.
2. We assume that the eigenvector corresponding to the most negative eigenvalue always points in the same direction with respect to the inertial reference frame in the non-comoving coordinate system.

First we define S_0 as the inertial reference frame centered on the CM of the external galaxy groups listed in Table 1. We introduce also a second reference frame S_1 , non-inertial and comoving with the CM of the LG, approximately coincident with the CM of the pair M31 and MW. Moreover, we assume that this reference frame has its principal axis always collinear with the principal axis of the external tidal tensor, i.e. this frame is an eigen-system of the external tidal tensor. \mathbf{x}_{M31} is the coordinate vector

of M31 in S_0 , \mathbf{x}_{MW} is the coordinate vector of MW in S_0 , \mathbf{x} is the vector of a generic dwarf galaxy in S_0 and \mathbf{R} is the position of the M31-MW barycenter, i.e. the origin of S_1 in S_0 . The generic position vector in the frame S_1 is $\boldsymbol{\xi}$. It will be used to indicate the generic position of a dwarf galaxy. Both reference frames are orthogonal (see Fig. 1).

Equations of motion. We only point out here that the generic position vectors in the two reference frames are now written as $\mathbf{x} = \mathbf{x}_{\text{LG}}^{\text{CM}} + \mathbf{O}\boldsymbol{\xi} = \mathbf{R} + \mathbf{O}\boldsymbol{\xi}$, where $\mathbf{O} \in SO(3)$ is the generic rotation matrix with elements O_{ij} . Similarly, for the time derivative of \mathbf{x} we can write $\dot{\mathbf{x}} = \dot{\mathbf{x}}_{\text{LG}}^{\text{CM}} + \mathbf{O}(\dot{\boldsymbol{\xi}} + \boldsymbol{\Omega} \times \boldsymbol{\xi})$ or, in more compact notation

$$\dot{\mathbf{x}} = \mathbf{V} + \mathbf{O}(\dot{\boldsymbol{\xi}} + \boldsymbol{\Omega} \times \boldsymbol{\xi}),$$

where $\boldsymbol{\Omega}$ is the angular velocity and we put $\mathbf{V} = \dot{\mathbf{x}}_{\text{LG}}^{\text{CM}}$.

The time derivative of the velocity equation yields the equation of motion

$$\ddot{\mathbf{x}} = \mathbf{A} + \mathbf{O}(\ddot{\boldsymbol{\xi}} + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{\xi}} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{\xi} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\xi})),$$

where, in the second equation the vector \mathbf{A} has been used to simplify the notation for the acceleration $\ddot{\mathbf{x}}_{\text{LG}}^{\text{CM}}$ of the barycenter. Moreover, to better underline the physical meaning of the different terms we recollect the previous equation as

$$\ddot{\mathbf{x}} = \mathbf{O}^T(\ddot{\mathbf{x}} - \mathbf{A}) - 2\boldsymbol{\Omega} \times \dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\Omega}} \times \boldsymbol{\xi} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\xi}). \quad (3)$$

From this last equation it is quickly evident that the acceleration $\ddot{\boldsymbol{\xi}}$, suffered by a dwarf galaxy in the non-inertial reference frame, is the sum of different terms due to the motion of the frames S_1 and S_0 and the fictional forces that appear thanks to the non-inertial nature of S_1 : $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\xi})$ is the centrifugal effect, $-2\boldsymbol{\Omega} \times \dot{\boldsymbol{\xi}}$ the Coriolis effect proportional to the velocity of the dwarf, and $-\dot{\boldsymbol{\Omega}} \times \boldsymbol{\xi}$ is the effect caused by the non-constant rotation rate of S_1 . As usual we assume that the rotation matrix linking the different orthonormal reference frames is $\mathbf{O} : \mathbf{O}\mathbf{O}^T = \mathbf{I}$ where the T stands for transpose and \mathbf{I} is the identity matrix.

From Eqn. (3) it is easily evident how to evaluate the term which gives the acceleration of a generic dwarf, $\ddot{\mathbf{x}} - \mathbf{A}$. Now the acceleration of a dwarf galaxy in S_0 is due to three contributions: the gradient in the external potential $\nabla\Phi_{\text{ext}}$, the gradient in the MW potential $\nabla\Phi_{\text{MW}}$, and the gradient in the M31 potential, $\nabla\Phi_{\text{M31}}$, i.e.

$$\ddot{\mathbf{x}} = -\nabla_{\mathbf{x}}(\Phi_{\text{ext}} + \Phi_{\text{MW}} + \Phi_{\text{M31}}). \quad (4)$$

Taylor expanding Φ_{ext} at the second order around the comoving barycenter, and with the usual definition of the Tidal Tensor given in equation (1), we can easily write

$$\Phi_{\text{ext}}(\mathbf{x}) \simeq \Phi_{\text{ext}}(\mathbf{R}) + \langle \nabla_{\mathbf{R}}\Phi_{\text{ext}}(\mathbf{R}), \mathbf{x} - \mathbf{R} \rangle - \frac{1}{2} \langle \mathbf{T}_{\text{ext}}(\mathbf{R})(\mathbf{x} - \mathbf{R}), \mathbf{x} - \mathbf{R} \rangle \quad (5)$$

together with two other similar relations for the potentials of MW and M31 that are not given here for the sake of brevity, with $\langle \cdot, \cdot \rangle$ expressing the standard inner product. Inserting these three relations in Eqn. (4), after some simplifications we obtain the acceleration $\ddot{\mathbf{x}} - \mathbf{A}$ in linear approximation:

$$\ddot{\mathbf{x}} - \mathbf{A} \simeq \langle \mathbf{T}_{\text{ext}}(\mathbf{R}) + \mathbf{T}_{\text{MW}}(\mathbf{R}) + \mathbf{T}_{\text{M31}}(\mathbf{R}), \mathbf{x} - \mathbf{R} \rangle.$$

By introducing the TTT as defined in the preceding section with

$$\mathbf{T}(\mathbf{R}) = \mathbf{T}_{\text{ext}}(\mathbf{R}) + \mathbf{T}_{\text{MW}}(\mathbf{R}) + \mathbf{T}_{\text{M31}}(\mathbf{R})$$

and using $\mathbf{x} - \mathbf{R} = \mathbf{O}\boldsymbol{\xi}$, we can simplify Eqn. (3) that becomes

$$\ddot{\boldsymbol{\xi}} = \mathbf{O}^T \mathbf{T} \mathbf{O} \boldsymbol{\xi} - 2\boldsymbol{\Omega} \times \dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\Omega}} \times \boldsymbol{\xi} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\xi}), \quad (6)$$

where we have dropped the explicit dependence on the position of the tidal tensor \mathbf{T} . This tensor is always evaluated at the barycenter $\mathbf{T} = \mathbf{T}(\mathbf{R})$ if not specified otherwise.

3.1. Energy states of equilibrium

Our aim now is to understand whether the equation of motion (6) can lead to stable equilibrium configurations. There are several techniques for attacking this problem based on the integration over a time interval of the force or the impulse, see e.g. (Binney & Tremaine 1987, Chap 7), the elegant Lagrangian treatment of Gnedin et al. (1999), or the sophisticated analysis in the action space of Weinberg (1994a,b).

To proceed further we look for the energy equilibrium configurations and their stability of a system governed by the equations of motion (6). Here, we take advantage of the fact that we can follow the evolution of the angular velocity of the non-inertial reference frame by looking at the motion on the sky of the eigenvector of the tidal tensor as already made in previous sections for the quadrupole, see Fig. 2.

It can be demonstrated that the Lagrangian leading to Eqn. (6), up to a total derivative, can be written as

$$L = \frac{m}{2} \|\dot{\boldsymbol{\xi}}\|^2 + m \langle \dot{\boldsymbol{\xi}}, \boldsymbol{\Omega} \times \boldsymbol{\xi} \rangle + \frac{m}{2} \|\boldsymbol{\Omega} \times \boldsymbol{\xi}\|^2 - m \langle \mathbf{A}, \mathbf{O}\boldsymbol{\xi} \rangle - W$$

(see for instance Landau & Lifshitz 1969). Therefore, taking the linear momentum $\mathbf{p} = \frac{\partial L}{\partial \dot{\boldsymbol{\xi}}} = m\dot{\boldsymbol{\xi}} + m(\boldsymbol{\Omega} \times \boldsymbol{\xi})$ we can evaluate the energy $E = \langle \mathbf{p}, \dot{\boldsymbol{\xi}} \rangle - L$ as

$$E = \frac{m}{2} \|\dot{\boldsymbol{\xi}}\|^2 - \frac{m}{2} \|\boldsymbol{\Omega} \times \boldsymbol{\xi}\|^2 + m \langle \mathbf{A}, \mathbf{O}\boldsymbol{\xi} \rangle + W.$$

Taking the derivative with respect to the positions, we obtain

$$\frac{\partial E}{\partial \boldsymbol{\xi}} = m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\xi}) - m\mathbf{O}^T \mathbf{T} \mathbf{O} \boldsymbol{\xi}. \quad (7)$$

The equilibrium energy is then given by the solution of the equation

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\xi}) - \mathbf{O}^T \mathbf{T} \mathbf{O} \boldsymbol{\xi} = 0. \quad (8)$$

In the reference frame S_1 , by definition the external tidal tensor is always diagonal, i.e. $\mathbf{O}^T \mathbf{T}_{\text{ext}}(\mathbf{R}) \mathbf{O}$ is greatly simplified but, in contrast, the first term of Eqn. (8) has a complicated structure and the same occurs for the other two terms composing \mathbf{T} . This way of proceeding does not allow for significant simplifications but a better insight can be gained by moving to a new reference frame, S_2 , with the same origin as the previous reference system S_1 , in which the matrices representing the tidal tensors \mathbf{T}_{M31} and \mathbf{T}_{MW} are both in diagonal form, but where \mathbf{T}_{ext} is not. From the analysis of the orbit evolution expected in a statistical interpretation of the Minimum Action, PC07 showed that MW and M31 are roughly coplanar to the external force field, at least during the last 9 Gyr. Therefore, in our simple model we can assume that the angular velocity vector $\boldsymbol{\Omega}_{\xi_3}$ with which we describe the rotation of the system of reference S_1 , which we remember here again for sake of clarity has its third axis ξ_3 parallel to the normal $\hat{\mathbf{n}}(\pi)$, has to be parallel to the angular velocity vector in this new system of reference $(O, \hat{\mathbf{e}}_{\eta_1}, \hat{\mathbf{e}}_{\eta_2}, \hat{\mathbf{e}}_{\eta_3})$, namely $\boldsymbol{\Omega}_{\eta_3}(S_2)$, that we assume rotating around its third axis η_3 , i.e. $\boldsymbol{\Omega}_{\eta_3}(S_2) \parallel \boldsymbol{\Omega}_{\xi_3}(S_1)$ over the last 9 Gyr. This means that we do not allow the orbital plane of M31 and MW to tilt with respect to the plane $\tau_{\lambda_3} \cong \pi$ orthogonal to the eigenvector corresponding to the most negative eigenvalue $\lambda_3(t)$ in the course of time evolution. Even though this assumption is automatically fulfilled in the analysis below, before looking at the evolution of the MW and M31 in the sky centered on the LG barycenter, we have checked that the angular distance between the position vectors of MW or M31 and $\hat{\mathbf{n}}(\tau_{\lambda_3})$ is $90^\circ \pm 3^\circ$, thus indirectly showing that the propagation of numerical errors in the calculation of the temporal evolution of the eigenvectors is small.

On the basis of these considerations, we can suppose that $\exists \mathbf{N} = \mathbf{N}(t) : \boldsymbol{\xi} = \mathbf{N}\boldsymbol{\eta}$, i.e. there exists a linear operator represented by a rotational matrix $\mathbf{N} \in SO(3)$ such that the generic position vector of a dwarf galaxy in S_1 , $\boldsymbol{\xi}$, can be written as a function of the generic position vector in S_2 , $\boldsymbol{\eta}$, and the tidal tensor of the external potential can be written as:

$$\begin{aligned} \mathbf{O}^T \mathbf{T}_{\text{ext}} \mathbf{O} \mathbf{N} \boldsymbol{\eta} &= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} N_{11} & N_{12} & 0 \\ N_{21} & N_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \\ &= \begin{pmatrix} N_{11}\eta_1\lambda_1 + N_{12}\eta_2\lambda_1 \\ N_{21}\eta_1\lambda_2 + N_{22}\eta_2\lambda_2 \\ \eta_3\lambda_3 \end{pmatrix} \end{aligned}$$

where we have adopted a rotation matrix spinning about $\boldsymbol{\Omega}_{\eta_3}$. As we can see, no explicit dependence on the rotational coefficients of the matrix \mathbf{O} is necessary in this reference system S_2 that we have chosen but the reader should keep in mind that $\mathbf{O}(t) \neq \mathbf{N}(t) \forall t$ and only in this new reference system S_2 have we been able to simplify the matrix as above. The general form of the rotational matrix could be casted as a combination of trigonometric functions, but this would be superfluous here. The only important thing to note is that even if the assumption of collinearity between the axis of rotation of S_1 and S_2 has

been justified, nothing can be said about the moduli of their angular velocities. We *cannot* assume that the rotation of the pair MW and M31 or, equivalently, the reference frame tightened to this rotation spins with the same rotational velocity of the reference frame attached to the external potential! This could lead to wrong or paradoxical results that need to be avoided. In other words we cannot assume $\|\boldsymbol{\Omega}_{\eta_3}(S_2)\| = \|\boldsymbol{\Omega}_{\xi_3}(S_1)\|$, which is wrong, but simply adopt the condition on the directions

$$\frac{\boldsymbol{\Omega}_{\eta_3}(S_2)}{\|\boldsymbol{\Omega}_{\eta_3}(S_2)\|} = \frac{\boldsymbol{\Omega}_{\xi_3}(S_1)}{\|\boldsymbol{\Omega}_{\xi_3}(S_1)\|}.$$

For example we can explicitly write the matrix \mathbf{N} as

$$\begin{pmatrix} N_{11} & N_{12} & 0 \\ N_{21} & N_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\gamma = \gamma(t)$ is the angle between the two systems. It will change as a function of time due to the time dependence of the angular velocity of the two frames

$$\gamma = \gamma\left(\Omega_{\xi_3}^{S_1}(t), \Omega_{\eta_3}^{S_2}(t)\right).$$

This also means that the angular velocity is not related to the angular momentum in a simply way. It will indeed be the result of the combined action of the centrifugal force due to the rotation of the frame tightened to the external force field, i.e. \mathbf{T}_{ext} , and the centrifugal force due to the rotation of the frame tightened to the motion of M31 and MW via \mathbf{T}_{MW} and \mathbf{T}_{M31} . We have a double centrifugal effect: one caused by the external field and the other by the MW and M31 that acts with different characteristic angular velocity (typically time-dependent angular velocities). The intensity of the force due to the centrifugal component can be written as

$$\begin{aligned} &\|\mathbf{N}\boldsymbol{\Omega}(S_1)\|^2 \|\mathbf{N}\boldsymbol{\eta}\| + \|\boldsymbol{\Omega}(S_2)\|^2 \|\boldsymbol{\eta}\| = \\ &= \|\boldsymbol{\Omega}(S_1)\|^2 \|\boldsymbol{\eta}\| + \|\boldsymbol{\Omega}(S_2)\|^2 \|\boldsymbol{\eta}\| = \\ &= \left(\|\boldsymbol{\Omega}(S_1)\|^2 + \|\boldsymbol{\Omega}(S_1)\|^2\right) \|\boldsymbol{\eta}\|, \end{aligned}$$

where $\|\boldsymbol{\eta}\|$ is the distance of the dwarf galaxy from the moving barycenter of S_2 . Moreover in this frame of reference S_2 we have that $\mathbf{O}^T \mathbf{T}_{\text{MW}} \mathbf{O} \mathbf{N} \boldsymbol{\eta}$ and $\mathbf{O}^T \mathbf{T}_{\text{M31}} \mathbf{O} \mathbf{N} \boldsymbol{\eta}$ are diagonal. Therefore, we immediately have (see e.g. Misner et al. (1973), Chap 1)

$$\mathbf{O}^T \mathbf{T}_{\text{MW}} \mathbf{O} \mathbf{N} \boldsymbol{\eta} = \frac{GM_{\text{MW}}}{\|\boldsymbol{\eta}_{\text{MW}}\|^3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

where for simplicity we have assumed that $\hat{\mathbf{e}}_{\eta_2}$ points from the barycenter of the system toward MW, $\hat{\mathbf{e}}_{\eta_3}$ is parallel to $\boldsymbol{\Omega}_{\xi_2}(S_1)$, and with the third axis oriented in such a way to form a left-handed reference frame. In the same way we may write the analogous for $\mathbf{O}^T \mathbf{T}_{\text{M31}} \mathbf{O} \mathbf{N} \boldsymbol{\eta}$. Finally, the tidal term in the condition for the energy equilibrium,

Eqn. (8), is given by

$$\mathbf{O}^T \mathbf{T} \mathbf{O} = \begin{pmatrix} \alpha_{11} & N_{12}\lambda_1 & 0 \\ N_{21}\lambda_2 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix}$$

with

$$\begin{aligned} \alpha_{11} &= N_{11}\lambda_1 - \frac{GM_{\text{MW}}}{\|\boldsymbol{\eta}_{\text{MW}}\|^3} - \frac{GM_{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3} \\ \alpha_{22} &= N_{22}\lambda_2 + \frac{2GM_{\text{MW}}}{\|\boldsymbol{\eta}_{\text{MW}}\|^3} + \frac{2GM_{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3} \\ \alpha_{33} &= \lambda_3 - \frac{GM_{\text{MW}}}{\|\boldsymbol{\eta}_{\text{MW}}\|^3} - \frac{GM_{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3} \end{aligned}$$

which provides a simple description of the tidal effects. The above relation can be further simplified by recalling that, from the definition of barycenter in the reference frame in use, we can write

$$\eta_2^{\text{MW}} = -\frac{\eta_2^{\text{M31}} M_{\text{M31}}}{M_{\text{MW}}}. \quad (9)$$

Inserting this expression into Eqn. (8), written in the S_2 frame, and using the relation $\|\boldsymbol{\eta}_{\text{MW}}\| = \sqrt{0 + (\eta_2^{\text{MW}})^2 + 0} = |\eta_2^{\text{MW}}|$ i.e. $\|\boldsymbol{\eta}_{\text{MW}}\|^3 = |\eta_2^{\text{MW}}|^3$, after tedious algebraic simplifications, we get for the three components of the tidal energy

$$\begin{aligned} \mathbf{O}^T \mathbf{T} \mathbf{O} \boldsymbol{\eta} &= \\ &= \begin{pmatrix} N_{12}\lambda_1\eta_2 + \eta_1 \left(N_{11}\lambda_1 - \frac{G(M_{\text{MW}}^4 + M_{\text{M31}}^4)}{M_{\text{M31}}^3 |\eta_2^{\text{M31}}|^3} \right) \\ N_{21}\lambda_2\eta_1 + \eta_2 \left(N_{22}\lambda_2 + \frac{2G(M_{\text{MW}}^4 + M_{\text{M31}}^4)}{M_{\text{M31}}^3 |\eta_2^{\text{M31}}|^3} \right) \\ \eta_3 \left(\lambda_3 - \frac{G(M_{\text{MW}}^4 + M_{\text{M31}}^4)}{M_{\text{M31}}^3 |\eta_2^{\text{M31}}|^3} \right) \end{pmatrix} \end{aligned}$$

In the same way we can derive the term due to the apparent force. Switching to the revolving system S_2 we apply another rotation expressed by $\mathbf{N}\boldsymbol{\Omega} \times (\mathbf{N}\boldsymbol{\Omega} \times \mathbf{N}\boldsymbol{\eta}) = \mathbf{N}(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\eta}))$ where $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{\Omega}(S_1), \boldsymbol{\Omega}(S_2))$ is the angular velocity of the system S_2 as seen from S_0 . The resulting centrifugal term is

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\eta}) = \begin{pmatrix} -(N_{11}\eta_1 + N_{12}\eta_2)\Omega_3^2 \\ -(N_{21}\eta_1 + N_{22}\eta_2)\Omega_3^2 \\ 0 \end{pmatrix}.$$

The associated equilibrium energy state is given by the solution of the systems

$$\begin{cases} -N_{12}\eta_2(\lambda_1 + \Omega_3^2) + \eta_1(\gamma - N_{11}\lambda_1 - N_{11}\Omega_3^2) = 0 \\ -2\gamma\eta_2 - (N_{21}\eta_1 + N_{22}\eta_2)(\lambda_2 + \Omega_3^2) = 0 \\ (\gamma - \lambda_3)\eta_3 = 0 \end{cases} \quad (10)$$

where

$$\gamma \equiv \frac{G(M_{\text{MW}}^4 + M_{\text{M31}}^4)}{M_{\text{M31}}^3 |\eta_2^{\text{M31}}|^3} > 0 \forall t. \quad (11)$$

The system (10) is the result we are looking for. It is evident from this system of equations that the plane $\eta_3 = 0$ is the equilibrium plane for the dynamical evolution of the gravitational system.

The above result is fully adequate for our purposes because the total potential is separated into the radial

and vertical components. This is a standard consequence of the linear approximation obtained by truncating the Taylor expansion of the potential at the second order. Equivalently one could use a generating function satisfying the Stakel theorem for separability in a Hamilton-Jacobi equation for the above system (e.g Boccaletti & Pucacco (1998)).

As far as the temporal evolution is concerned ($t \in [t_{\text{inf}}, t_0]$), we have $\eta_3(t) = 0$ which, translated into our spatial resolution, simply means a physical spatial resolution $|\eta_3| < 100 \text{ kpc}$. Furthermore, one can never have $(\gamma - \lambda_3) = 0$ because this would imply that $\exists t : \lambda_3(t) = \gamma(t)$, whereas according to definition (11) we have $\gamma(t) > 0 \forall t$, and finally $\lambda_3 < 0 \forall t_{\text{lb}} < 9 \text{ Gyr}$ as shown by Fig. 2. These last two conditions are *clearly inconsistent* leading to a contradiction that concludes our proof as required.

Therefore, the major conclusion of this demonstration is that the only possible solution is the following one: the statistical minimization of PC07 is compatible with a planar distribution of the dwarfs and it is not a mere coincidence. This result completes the missing interpretation of the result already included in PC07.

3.2. Stability of the equilibrium configuration

Finally, we examine the stability of the equilibrium plane that we have found by solving Eqn. (10). We take the energy of Eqn. (7) written for the system S_2

$$\frac{\partial E}{\partial \boldsymbol{\eta}} = m\mathbf{N}[\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\eta})] - m\mathbf{O}^T \mathbf{T} \mathbf{O} \mathbf{N} \boldsymbol{\eta}$$

and calculate the derivative

$$\frac{\partial^2 E}{\partial \boldsymbol{\eta} \partial \boldsymbol{\eta}} = m \frac{\partial}{\partial \eta_g} \left\{ N_{ij} \varepsilon_{jkl} \Omega_k \varepsilon_{lmn} \Omega_m \eta_n - [\mathbf{T}\mathbf{N}]_{ij} \eta_j \right\},$$

where ε_{ijk} is the Levi-Civita symbol, summation over the repeated index is assumed and we define $[\mathbf{T}\mathbf{N}]_{ij} \equiv [\mathbf{O}^T \mathbf{T} \mathbf{O} \mathbf{N}]_{ij}$. Indicating with $\mathbf{S} = \mathbf{O}\mathbf{N}$ the composition of the two rotation matrices \mathbf{O} from S_0 to S_1 and \mathbf{N} from S_1 to S_2 , after some algebraic simplifications we obtain a more compact form

$$\frac{\partial^2 E}{\partial \boldsymbol{\eta} \partial \boldsymbol{\eta}} = -m\mathbf{N} \left\{ \boldsymbol{\Theta} + \mathbf{S}^T \mathbf{T} \mathbf{S} \right\} \quad (12)$$

where we have defined another matrix

$$\begin{aligned} \Theta_{ij} &\equiv \|\boldsymbol{\Omega}\|^2 \delta_{ij} - \Omega_i \Omega_j = \\ &= \begin{pmatrix} \Omega_2^2 + \Omega_3^2 & -\Omega_1 \Omega_2 & -\Omega_1 \Omega_3 \\ -\Omega_2 \Omega_1 & \Omega_1^2 + \Omega_3^2 & -\Omega_2 \Omega_3 \\ -\Omega_3 \Omega_1 & -\Omega_3 \Omega_2 & \Omega_1^2 + \Omega_2^2 \end{pmatrix} \end{aligned}$$

This matrix has no inverse, determinant $|\Theta| = 0$, and trace $\text{Tr}(\Theta) = 2\|\boldsymbol{\Omega}\|^2$.

At this stage the usual procedure would be to solve for the eigen-system (12). However, we can avoid this complication by noticing that in the linear approximation the

vertical potential decouples from the radial one. In other words, applying the full procedure we would obtain three eigenvalues $\{\alpha_1(t), \alpha_2(t), \gamma - \lambda_3(t)\}$, where $\alpha_1(t)$ and $\alpha_2(t)$ are two complicated functions of the time, whereas the last eigenvalue has to be exactly the most negative eigenvalue of the external tidal field for which we have already calculated the time dependence as shown in Fig. 2. As long as the eigenvalue remains negative, the relation $\gamma(t) - \lambda_3(t) > 0 \Leftrightarrow \gamma(t) > \lambda_3(t)$ holds because $\gamma > 0$ by definition (equation (10)) whereas $\lambda_3 < 0$ as required in Fig. 2. **Therefore the plane π is stable.**

3.3. Equations of motion and force balance

Finally, we can get a much deeper insight for the physical reasons of the existence of the plane π by analyzing the equation of motion in S_2 . If the plane π is a stable configuration of the spatial distribution of dwarf galaxies, it is important to isolate the force acting on it. The equations of motion in S_2 are, from e.g. Eqn. (6),

$$\ddot{\boldsymbol{\eta}} = \mathbf{N}^T (\langle \mathbf{T}_{\text{ext}}(\mathbf{R}) + \mathbf{T}_{\text{MW}}(\mathbf{R}) + \mathbf{T}_{\text{M31}}(\mathbf{R}), \mathbf{N}\boldsymbol{\eta} \rangle) - 2\boldsymbol{\Omega} \times \dot{\boldsymbol{\eta}} - \dot{\boldsymbol{\Omega}} \times \boldsymbol{\eta} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\eta})$$

where for the different elements of the tidal tensor we can now write

$$\mathbf{N}^T \mathbf{T}_{\text{M31}} \mathbf{N} \boldsymbol{\eta} = \begin{pmatrix} -\frac{GM_{\text{M31}}\eta_1^{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3} & 0 & 0 \\ 0 & \frac{2GM_{\text{M31}}\eta_2^{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3} & 0 \\ 0 & 0 & -\frac{GM_{\text{M31}}\eta_3^{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3} \end{pmatrix},$$

an analogous equation for MW, and

$$\mathbf{N}^T \mathbf{T}_{\text{ext}} \mathbf{N} = \begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

The equilibrium in the π plane is given by

$$\begin{cases} \left(T_{11} - \frac{GM_{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3} - \frac{GM_{\text{MW}}}{\|\boldsymbol{\eta}_{\text{MW}}\|^3} \right) \eta_1 + T_{12}\eta_2 + \eta_1\Omega_3^2 - 2\Omega_3\dot{\eta}_2 + \dot{\Omega}_3\eta_2 = 0 \\ T_{12}\eta_1 + \left(T_{22} + \frac{2GM_{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3} + \frac{2GM_{\text{MW}}}{\|\boldsymbol{\eta}_{\text{MW}}\|^3} \right) \eta_2 + \eta_2\Omega_3^2 + 2\Omega_3\dot{\eta}_1 - \dot{\Omega}_3\eta_1 = 0 \\ \eta_3 \left(-\frac{GM_{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3} - \frac{GM_{\text{MW}}}{\|\boldsymbol{\eta}_{\text{MW}}\|^3} + \lambda_3 \right) = 0 \end{cases}$$

which sheds light on what is happening in reality³.

Along the direction orthogonal to the plane π , three forces are present. One is due to the MW, $-\frac{GM_{\text{MW}}}{\|\boldsymbol{\eta}_{\text{MW}}\|^3}$, another due to M31, $-\frac{GM_{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3}$, and the third one due to the external field $+\lambda_3 < 0$. They all sum together to push any dwarf which tends to escape from the position of equilibrium to return back in the plane. **Therefore they tend to**

flatten the whole system. This tendency of flattening the distribution of dwarf galaxies, and claiming the stability of the plane π , can be considered valid in the limit of the linear approximation, i.e. roughly for 150 kpc above and below the plane and for a period of time of roughly 9 Gyr, thus being partially able to imprint the initial proper motions of the dwarf galaxies taken into consideration for the Local Group.

Along the directions parallel to the plane, the situation is more complicated and described by relations like

$$\left(T_{11} - \frac{GM_{\text{M31}}}{\|\boldsymbol{\eta}_{\text{M31}}\|^3} - \frac{GM_{\text{MW}}}{\|\boldsymbol{\eta}_{\text{MW}}\|^3} \right) \eta_1 + T_{12}\eta_2 + \eta_1\Omega_3^2 - 2\Omega_3\dot{\eta}_2 + \dot{\Omega}_3\eta_2 = 0$$

The first two terms in the sum on the left hand side are due to the tidal field that has to balance the third term due to the centrifugal force, the last term is due to the Coriolis effect and the extra term deriving from a non-uniform rotation of the system S_2 . The same kind of relation holds for the other coordinates.

4. Summary, conclusions and consideration on the limits of the approach used

The spatial distribution of the galaxies in the LG is the footprint of its formation mechanism, the internal gravitational interactions among the galaxies, and the gravitational action of external massive galaxies or galaxy groups on the LG members.

In this paper, we have thoroughly addressed the whole subject focusing the attention on the role played by the tidal force field exerted by external galaxies or galaxy groups on the dwarf galaxies of the LG, excluding those that are clearly under the dominant gravitational effects of the HGs, in shaping the large-scale distribution of the LG galaxies.

The idea stands on the well known effect of tidal interactions, which can be expressed as a function of the gradient in the gravitational force. While the gravitational force never changes sign, its gradient can do so. Moreover, while the gravitational force field at any distance from the center of mass of a system depends only on the inner distribution of matter, the tidal force field does not; it is indeed the result of both internal and external distributions of matter. The tidal force acting on a body moving along a certain direction will pull it away from the origin of the reference frame and, at the same time, push it along directions perpendicular to the motion toward the origin of the reference frame. Therefore *a system subjected to tidal interactions can undergo not only tidal stripping but also tidal compression. In other words, the space distribution of galaxies undergoing tidal interactions tends to become flat.*

The results of this study can be summarized as follows:

- The tidal forces can be the cause of the planar distribution of these dwarf galaxies. We analytically obtain

³ We did not exploit here the Eqn. (9) previously necessary for the energy analysis, in favor of a clearer and easier physical interpretation of the terms in the equation.

the same numerical results of Pasetto & Chiosi (2007). In fact, we prove that a planar distribution of all dwarf galaxies, excluding those tightly bounded to a HG, is compatible with the presence of an external force field.

- The planar geometrical distribution found by Pasetto & Chiosi (2007) was not known to relate to the most negative eigenvalue (and associated eigenvector) of the tidal tensor. In that sense this previous work was partially incomplete. Here we have gone deeper into this issue following the original idea of Raychaudhury & Lynden-Bell (1989) to check whether the planar distribution is a mere coincidence or the consequence of fundamental laws of mechanics. To address this, we first check, using different arguments, the coincidence between the direction given by the vector orthogonal to the geometrical plane and that corresponding to the eigenvector with the most negative eigenvalue. Second, we analyze the energy of the orbital motion of the LG galaxies and find that the minimum energy corresponds to a planar distribution which is exactly the geometrical plane π . Therefore the planar distribution is the consequence of the long time-scale influence of the tidal forces exerted by massive galaxies or galaxy groups external to the LG. Finally, we demonstrate that this situation has been stable over the past 9 Gyr.
- The equilibrium and stability of the plane is a consequence of the minimum in the Action and of the orbits that come from this minimum. Although Pasetto & Chiosi (2007) have carefully investigated the nature of the Minimum Action (whether local or absolute), the uncertainty affecting the orbits derived from the Action minimization could lead to an uncertainty in the energy analysis, the stability of the plane and the compression effect in turn here evidenced. This problem is still unsolved and cannot be resolved at the present time because better observational data would be required (proper motions, velocities, distance moduli, absolute positions, masses, etc.). Hence, the analytical approach presented here still requires further support from independent arguments. Along this line of thought is the study of van der Marel & Guhathakurta (2008) who find compatibility between their results and what in Pasetto & Chiosi (2007).

The completeness of the sample of external galaxies listed in Table 1 is another factor influencing results developed here as well as in Pasetto & Chiosi (2007). In the previous paper special care was taken to confirm the results based on the minimization of the Action with an extended catalog from Peebles et al. (2001). Moreover, we can confirm an excellent concordance between the actual direction of the quadrupole tensor eigenvectors in Raychaudhury & Lynden-Bell (1989) and that derived here with the independent catalog compilation of Peebles et al. (2001). Nevertheless the continuous discoveries of new LG members suggest

that the actual census of the LG galaxies cannot yet be considered complete (e.g. Loeb & Narayan (2008)).

- The Minimum Action together with the study of the equilibrium through the first derivative of the energy of the system is a method that can be used to constrain the energy of dwarf galaxies with unknown proper motions. The missing proper motion prevents us from having the complete energy of any given dwarf, but the minimization of the action, together with the study of the tidal tensor, permits us to handle the derivative of the energy in this particular situation.
- We can find another example of a flat structure in the super-galactic plane, a slab of roughly $\cong 25$ Mpc in thickness and of a diameter greater than 110 Mpc (Lahav et al. 2000), which was already proposed by de Vaucouleurs in 1953 (de Vaucouleurs 1953). All the groups of galaxies used here (Table 1) lie within this plane. With respect to this super-galactic plane the normal of the plane π has direction $(SGL, SGB) = (95^\circ, 69^\circ)$. This plane has been determined requiring that the vector joining the MW and M31 explicitly belong to π . If you attempt to find a best fit determination for the plane, say $\tilde{\pi}$, of the overall sample of the LG galaxies, without forcing the MW and M31 to belong to such a plane, we have, from PC07, $(l, b) = (46^\circ, 29^\circ)$, i.e. $(SGL, SGB) = (93^\circ, 67^\circ)$, for the normal to the plane $\tilde{\pi}$. Thus, this is even closer to the direction of the vector associated with the most negative eigenvalue of the tidal tensor but slightly more distant from the direction of the super-galactic North Pole. The proximity of the direction of the normal to π and the super-galactic North Pole cannot yet be claimed as a significant result without further investigation. However, it is also not yet possible to study this problem fully, given the incompleteness of the catalogue of the nearby galaxies (see Karachentsev et al. (2004)). The mass estimation together with the distances for the systems involved are still the major source of the errors we considered. Finally, in none of the external groups taken into account in Table 1 can we claim the existence of a similar flat distribution of dwarf galaxies that could be a first indication of a common external effect acting on all these groups. It seems that the local dwarf galaxies are primarily influenced the external potential of nearby groups, not by the larger and more distant mass accumulation responsible for the super-galactic plane.
- The issue of the spatial distribution of nearby satellite galaxies bound to their HG remains to be addressed. These dwarf galaxies have orbits whose typical distance lies inside the dark matter halo of the hosting galaxy. Many N-body simulations have shown that galaxies in close proximity of a HG are subjected to strong collisional interactions. The satellite dwarf galaxies may suffer kick-off, interactions and may be absorbed into the halo of the HG (MW and M31 in our case), and may abruptly change the phase-space density distribution function. In other words, the

large-scale description of the gravitational interaction adopted here and in Pasetto & Chiosi (2007) does not work at the short distance scales of the HG-dwarf satellite interactions. This will be the subject of a forthcoming study, in which simulations of the interaction between a HG and its satellites will be investigated in the framework of strong collisional dynamics along the line of work already initiated by Pasetto et al. (2003) and Pasetto et al. (2009).

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